The Critical Region in Finite-Sized Systems

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Abstract The critical point in an infinite system is smeared out or "rounded" in a finite system into a "critical region." This region has some structure associated with it, which is difficult to study except by means of simulations, *i.e.*, Monte Carlo. I have begun a study of this structure in small two-dimensional Ising model systems, using the Markov property method.

There is still an area of critical phenomena that has not been nearly as thoroughly explored as it might be. The regions where the temperature is greater than the critical temperature and where the temperature is less than critical temperature in infinite size systems have by now been quite thoroughly studied. However, when the system is finite in size, we know that the thermodynamic quantities which would have been divergent in an infinite system are rounded over a critical region instead. There is some interesting structure in this region described by finite-size scaling functions. Other problems which are of the same character have recently come under investigation by Borgs *et al.*, Caracciolo *et al.*, and Kim *et al.* [1]. Some methods of scaling are much less sensitive to the system size than others. Put otherwise, we see better data collapse (independence of the system size) using some sets of variables rather than other sets which are asymptotically the same. The elucidation of which versions to use is of considerable practical importance, as one can get by with much smaller system sizes than would otherwise be required.

In this paper I will begin such a study, primarily for the two-dimensional Ising model on small system sizes. The method of computation of these results is the Markov property method [2]. I have results at various temperatures for the 2, 4, 6, 8 and 10 edged squares on a plane square lattice with periodic boundary conditions. In their study of the question of hyperscaling in the three dimensional Ising model, Baker and Kawashima [3], and Baker and Erpenbeck, and Kim [4] before them, noticed that the variable ξ_L/L where ξ_L is the correlation length computed from the system of size L is very useful in this regard. That quantity is zero in the limit $L \to \infty$ for temperatures above the critical point and infinite for temperatures below the critical point. Thus the infinite system limit where this quantity has any finite value is just the critical point, and so looking at the behavior of the system for fixed ξ_L/L in effect spreads out the critical point. We define the correlation length by,

$$\xi_L^{-2} = [2f(\Delta k,0,0) + 2f(0,\Delta k,0) + 2f(0,0,\Delta k) \\ - f(0,\Delta k,\Delta k) - f(\Delta k,0,\Delta k) - f(\Delta k,\Delta k,0)]/3, \quad (1)$$

where $\Delta k \equiv 2\pi/L$, $\chi = \chi(\vec{0})$ is the magnetic susceptibility, $\langle\ \rangle$ is the mean value over all the states of the system, weighted by the Gibbs factor, and

$$f(\vec{k}) \equiv 4\left(\sin^2\frac{k_x}{2} + \sin^2\frac{k_y}{2} + \sin^2\frac{k_z}{2}\right)\left(1 - \frac{\chi(\vec{k})}{\chi}\right)^{-1},$$

$$\chi(\vec{k}) \equiv \left\langle \left| \sum_{\vec{r}} \exp(-i\vec{k}\cdot\vec{r}) S_{\vec{r}} \right|^2 \right\rangle, \quad (2)$$

where $S_{\vec{r}}$ are the spin variables. In the region $T > T_c$, this definition of the correlation length is correct up to the second order in $1/L$. For $T < T_c$ we continue to use this definition. It is the analytic continuation (for finite L), and is *not to be confused with the proper low temperature quantities.* The same remark is true of the other thermodynamic quantities we use. If M is the total magnetization, then the Binder cumulant ratio U, and the renormalized coupling constant g are given by,

$$U = \frac{\langle M^4 \rangle}{(\langle M^2 \rangle)^2} - 3, \quad g = -\left(\frac{L}{\xi_L}\right)^d U, \quad (3)$$

where d is the spatial dimension.

Next I remark that the argument usually used in finite scaling discussions instead of ξ_L/L is $L^{1/\nu}(K_c - K)$, where $K = J/kT$ with J the exchange energy and k Boltzmann's constant. (For the 2-D Ising model, $\nu = 1$.) There is a considerable difference in these two variables for small values of L as is shown in Fig. 1.

The point of the following graphs is that they are appropriate for quantities which differ on the high and low sides of the the critical point. They will show how the transition occurs. The critical point for an infinite system is a point of non-uniform approach in the system-size, temperature plane. The functions described in these graphs are basically finite-size scaling functions. It has been argued by Privman and Fisher [5] that they should be universal, at least in the

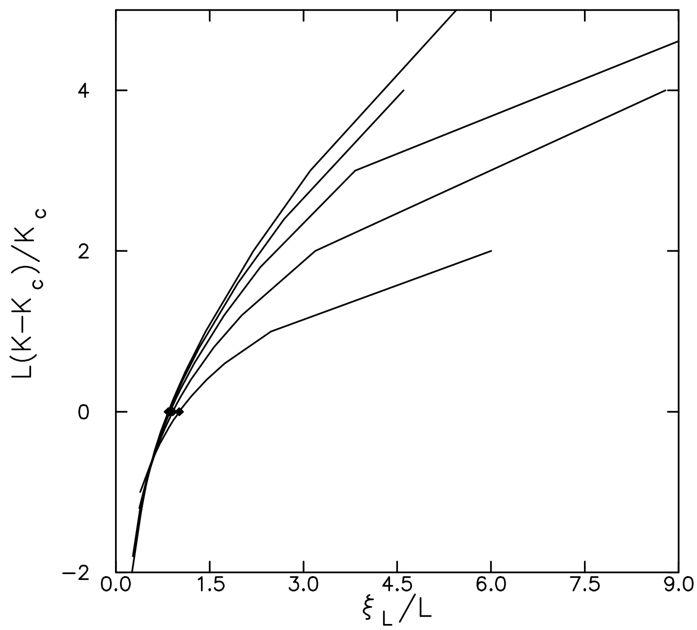

Fig. 1. $L(K_c - K)/K_c$ versus ξ_L/L for the 2D Ising model. The curves from left to right are for $L = 10, \ldots 2$, and the diamonds mark the value at the critical temperature.

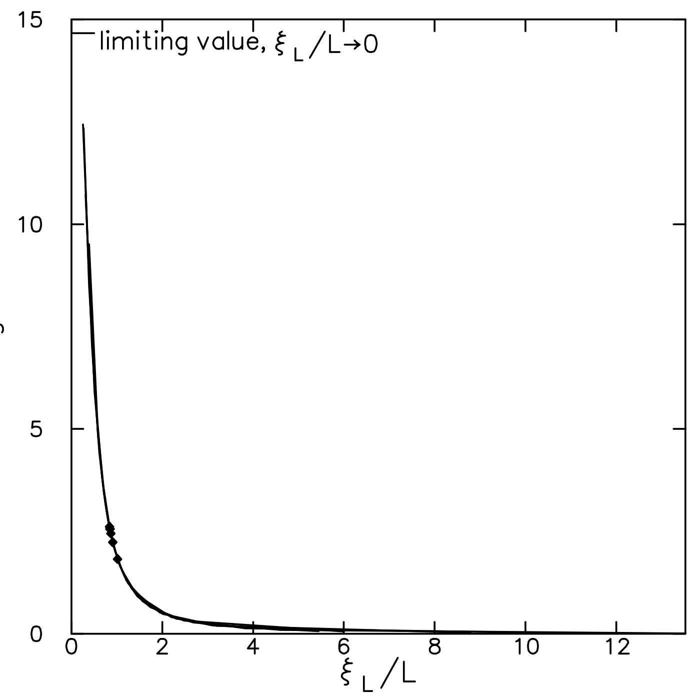

Fig. 2. g versus ξ_L/L for the 2-D Ising model. The curves are for $L = 10, \ldots 2$, and the diamonds mark the value at the critical temperature, with $L = 10$ the highest diamond.

high-temperature region. The most successful data collapse that I have found so far is for the renormalized coupling constant. These results are shown in Fig. 2. The limiting value for $\xi_L/L \to 0$ is marked by a large tic mark on the left hand margin. Note is taken that this limit is not the same as the value at $K = 0$, but rather is the limit obtained by first holding $\xi_L/L > 0$ fixed and taking $L \to \infty$ followed by the limit $\xi_L/L \to 0$. Note is taken that as the Binder cumulant ratio differs from g by a factor of $(\xi_L/L)^d$, a similar data collapse is also found for it as well.

Similar results were obtained by Baker and Kawashima [3] for the three dimensional Ising model. We show them in Fig. 3.

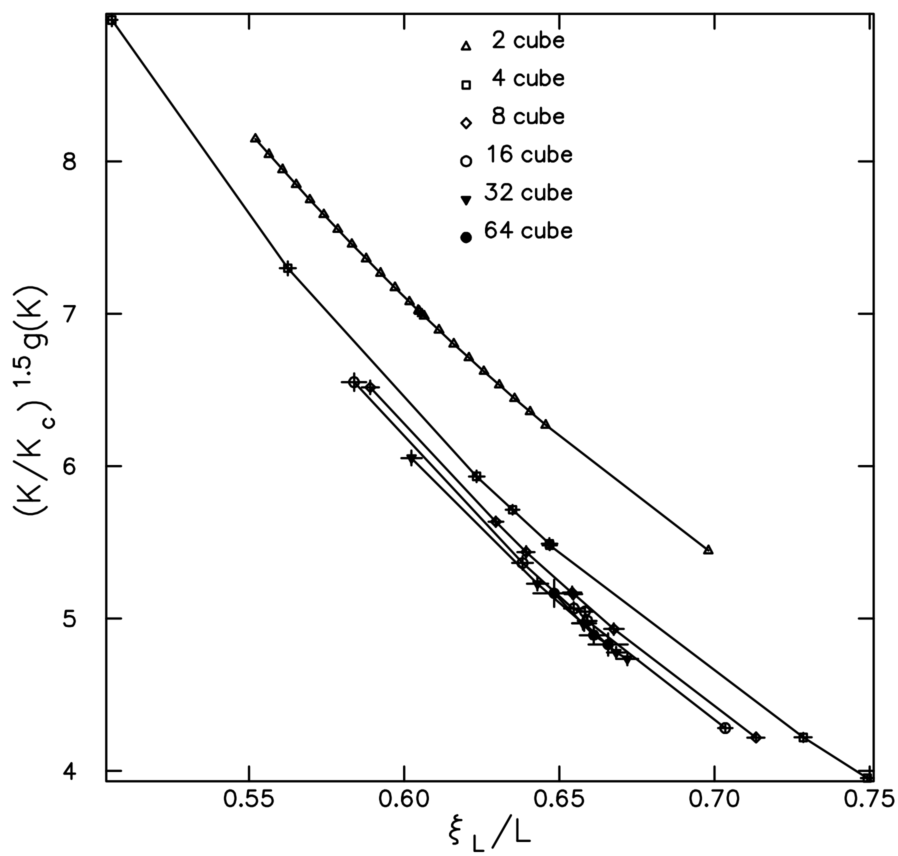

Fig. 3. $(K/K_c)^{3/2}g(K)$ versus ξ_L/L for the 3-D Ising model. The curves are for cubes with edge length $L = 2,\ 4,\ 8,\ 16,\ 32$, and 64, with periodic boundary conditions.

A further example is shown in Fig. 4. Here we plot $(K/K_c)^{7/8}\chi/\xi_L^{7/4}$. The factor of $\xi^{-7/4}$ reflects the standard scaling behavior in the two dimensional Ising model, and is chosen so the plotted quantity should be finite at the critical point. The factor of $(K/K_c)^{7/8}$ is included to compensate for the fact that $\xi \propto K^{1/2}$ as $K \to 0$. Thus the plotted quantity will also be finite at $K = 0$. Again the limit $\xi_L/L \to 0$ is marked with a large tic mark on the left margin. The same remarks about this limit apply as those made with respect to Fig.3.

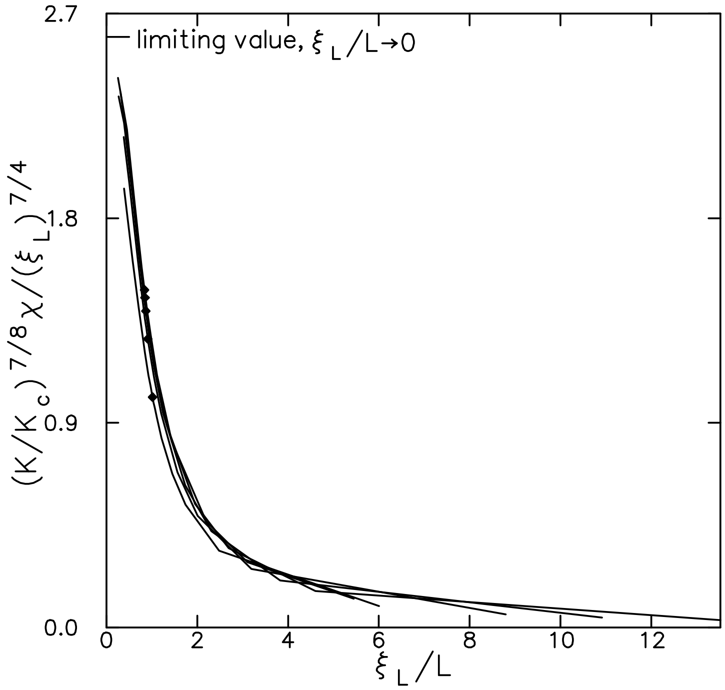

Fig. 4. $(K/K_c)^{7/8}g(K)/\xi_L^{7/4}$ versus ξ_L/L for the 2-D Ising model. The curves from left to right are for squares with edge lengths $L = 2, \ldots, 10$. The diamonds mark the values for the critical temperature, with $L = 10$ having the largest value.

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